

Statistical hadronization and hadronic microcanonical ensemble I

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Abstract. We present a full treatment of the microcanonical ensemble of the ideal hadron-resonance gas starting from a quantum-mechanical formulation which is appropriate for the statistical model of hadronization. By using a suitable transition operator for hadronization we are able to recover the results of the statistical theory, particularly the expressions of the rates of different channels. Explicit formulae are obtained for the phase space volume or density of states of the ideal relativistic gas in quantum statistics as a cluster decomposition, generalizing previous ones in the literature. The problem of the computation of averages in the hadron gas microcanonical ensemble and the comparison with canonical ones will be the main subject of a forthcoming second paper.

1 Introduction

The revived interest in the statistical model of hadron production is mainly owing to its application to heavy ion collisions, where an equilibrated source of hadrons is expected. This model has given strikingly good results in elementary collisions as well [1], and this finding has triggered some debate about their interpretation [2]. A proposed model is that hadronization occurs at some critical energy density [3] (or maybe another related parameter) of a number of massive pre-hadronic colourless extended objects (henceforth referred to as *clusters*) which are formed as a result of the underlying non-perturbative strong-interaction dynamics and which thereafter decay coherently into multihadronic states [4]. In this scheme, the single cluster's decay rate into any channel would be determined only by its phase space with no special dynamical weight (*phase space dominance*). Thereby, the observed statistical equilibrium would not be the effect of a collisional thermalization process between formed hadrons over long-lived extended regions in the final state, rather of equal quantum transition probabilities from a cluster to all accessible final states. By accessible it is meant that one must comprise only those states that fulfill conservation laws, i.e. have the same quantum numbers as the initial cluster's ones. The set of states with fixed energy-momentum, angular momenta, parity and internal charges is defined as *microcanonical ensemble*, though the same name is usually employed to denote the set of states with fixed energy-momentum and internal charges, relaxing angular momentum and parity conservation. We will not make any distinction either; whether the constraints of angular momenta and parity are meant to be included will be clear from the context.

Although the microcanonical ensemble is the correct statistical ensemble to use in hadronizing a single cluster, so far all actual data analyses within the statistical model have

been carried out in the framework of the canonical or grand-canonical ensemble of the hadron gas, i.e. with hadronizing sources described in terms of a temperature and taking into account the conservation of energy-momentum and angular momentum only on average. The microcanonical ensemble has been used very seldom [5,6], mainly owing to the hard and long computations involved. Indeed, in high energy collisions, where many clusters are produced, the use of the canonical ensemble is favored by fluctuations of masses and volumes, which tend to reduce the importance of exact conservation of energy and momentum. It is even possible that fluctuations make the system of many clusters equivalent (as far as Lorentz invariant quantities, such as average multiplicities, are concerned) to a large global cluster obtained by ideally clumping them [7]. While the canonical ensemble is in fact an increasingly better approximation of the microcanonical one for large values of cluster's mass and volume, we have no quantitative estimate of how large they ought to be¹, so the use of the canonical ensemble is, until now, justified by the agreement with the data. On the other hand, it would be desirable to have a more precise and quantitative assessment of the accuracy of the canonical approximation. An explicit calculation of the microcanonical ensemble of the hadron gas is also necessary if we want to test the statistical model at lower energies (say $\sqrt{s} < 10$ GeV), where conservation laws are expected to play a major role and the canonical approximation is not a good one. Furthermore, it would be very useful having at our disposal a Monte-Carlo algorithm for microcanonical hadronization of single clusters in high energy collisions to be used for numerical calculations of quantities for which an analytical expression cannot be obtained. Thus, providing a reliable and fast numerical

¹ Recently, a calculation has been done for *pp* collisions with a restricted set of hadrons [6]

algorithm for the calculation of the microcanonical ensemble of the hadron gas and comparing the results with the canonical approximations are the main goals of this work, which will be described in two papers.

In this first paper, we will confine ourselves to the analytical development of the microcanonical formalism, while the numerical calculations will be the main subject of the second paper. In fact, another major motivation of this work is to provide a consistent formulation of the statistical hadronization model starting from quantum transition probabilities, which is still lacking despite the apparent simplicity of the picture and the fact that its foundations were laid more than 50 years ago [8]. The attempts to derive the statistical theory results from S -matrix under suitable hypotheses, were mainly carried out by Hagedorn [9] and Cerulus [10] on the basis of time-reversal arguments. However, the reasoning is quite involved in this approach, and one needs a separate treatment of the dynamical matrix element averaging for multiplicities and spectra. The inclusion of quantum statistical effects in the microcanonical ensemble of the relativistic hadron gas has been done [11] consistently only for large volumes and not sufficiently general. In fact, in this traditional treatment, particle states confined in the cluster's volume are assumed to be eigenstates of energy-momentum, which is true only if the volume is so large that the entailed energy-momentum uncertainty can be neglected, which is not generally the case when dealing with small volumes (this will be discussed more in detail at the end of Sect. 2). Furthermore, in that approach, the whole treatment did not start from the statistical theory of multiple production, and it is thus not easy to generalize if angular momentum and parity conservation are to be included.

Therefore, we believe that a coherent general reformulation of the statistical model of hadronization is needed. In this paper, we will go along the whole formalism starting from the basic assumptions and will recover some well known formulae in the literature, like those in [11], as approximations of more general ones in case of sufficiently large volumes. In particular, we will recover the N -body relativistic phase space expression (without angular momentum and parity conservation) without treating confined particle states as energy-momentum eigenstates, an assumption which is correct only asymptotically. We will show how conservation laws are to be implemented in the most general case, thus providing a usable framework to obtain a more general expression of N -body phase space when angular momentum and other conserved quantities are to be taken into account. Furthermore, we will explicitly show how the microcanonical ensemble reduces to the canonical one for large cluster's volume and mass. The formulae presented in this work will be then the basis of the numerical computations in the second paper [12].

This paper is organized as follows: in Sect. 2 we will introduce a basic formulation of the statistical hadronization model and follow the path leading to the microcanonical ensemble; in Sect. 3 we will develop in detail the microcanonical formalism for an ideal hadron-resonance gas with full quantum statistics; in Sect. 4 the microcanonical partition

function will be calculated and the approximations needed to obtain closed expressions stressed, while in Sect. 5 the transition from the microcanonical to the canonical ensemble is described; finally, in Sect. 6 the calculation of the physical observables will be discussed.

2 Statistical hadronization of a cluster

The fundamental assumption of the statistical hadronization model is that the final stage of a high energy collision results in the formation of a set of extended colourless massive objects, the *clusters* or *fireballs*, producing hadrons in a purely statistical manner: that is, all multihadronic states within the cluster volume and compatible with the cluster's quantum numbers are equally likely. Clusters can indeed be thought of as very short-lived extended resonances, much like bags of the bag model [13]. They differ from the clusters proposed in other hadronization models [14] as they are endowed with a spatial extension. In this picture, the cluster's decay rate into a given N -particle channel should be proportional to the number of multiparticle states within the volume V of the cluster, which can be expressed, in the limit of Boltzmann statistics and neglecting angular momentum and parity conservation, as

$$\Gamma \propto V^N \int d^3\mathbf{p}_1 \dots d^3\mathbf{p}_N \delta^4 \left(P - \sum_{i=1}^N p_i \right), \quad (1)$$

where P is the four-momentum of the cluster. The distinctive feature of the statistical model is essentially the appearance of a finite volume in the decay rate, which makes the above expression different from that of the decay rate of a massive particle usually found in textbooks, where asymptotic states are defined over an infinitely large volume:

$$\Gamma \propto \int \frac{d^3\mathbf{p}_1}{2\epsilon_1} \dots \frac{d^3\mathbf{p}_N}{2\epsilon_N} |M_{if}|^2 \delta^4 \left(P - \sum_{i=1}^N p_i \right). \quad (2)$$

While these two equations have in principle a different physical meaning, there have been several attempts to derive an equation like (1) from (2) (see e.g. [9]). Instead of establishing a link between them through a suitable choice of the squared matrix element $|M_{if}|^2$, we will try to obtain the formula (1) starting from a suitable ansatz which will enable us also to recover quantum statistics effects (Bose–Einstein or Fermi–Dirac correlations) in a natural way. This can be accomplished because a finite volume is involved in (1) unlike in (2).

We assume that, as a result of non-perturbative QCD-driven evolution, the cluster state develops uniform projections over the multihadronic Fock space states defined by its volume and compatible with its quantum numbers. Thus, if $|i\rangle$ is a properly normalized asymptotic state characterized by the mass, spin and quantum numbers of the cluster and where $\langle f|$ is an asymptotic multihadronic final state, the rate Γ_f into the final state f is written as

$$\Gamma_f = |\langle f|W|i\rangle|^2, \quad (3)$$

where W is an effective transition operator proportional to the projector over the Hilbert subspace defined by all stationary multihadronic states $|h_V\rangle$ within the cluster, namely

$$W = \sum_{h_V} |h_V\rangle \langle h_V| \hat{\eta} \equiv P_V \hat{\eta}, \quad (4)$$

where $\hat{\eta}$ is an operator depending on the strong-interaction symmetry group invariants (Casimir operators) such as mass, spin, isospin, charge etc. The state $|h_V\rangle$ will be assumed to be a confined stationary free particle state within the cluster, with fixed or periodic boundary conditions; the inclusion of all resonances as independent states allows one to take into account a part of the interaction between strongly stable hadrons and this is the reason of the usual expression *ideal hadron-resonance gas* [15].

The operator W is a peculiar one because it is dependent on the shape and volume of the cluster, which in fact pertains to the initial conditions. If the cluster's quantum numbers coincide with those of the initial colliding system, (only one cluster is produced) W should commute with all conserved quantities in the strong interaction to ensure the due selection rules, though this may not be necessary if many clusters are produced.

The commutation requirement is fulfilled for all internal symmetries, charge conjugation and for angular momentum and parity provided that the cluster has a spherical shape (see Appendix A). On the other hand, W does not commute with energy and momentum, as translational symmetry is broken by the assumption of a finite volume; hence, a violation of energy-momentum conservation of the order of the inverse of the cluster's linear size is implied. However, as it will become clear in the following, momentum-integrated rates in fact get a contribution only from states fulfilling energy-momentum conservation; otherwise stated, a finite volume introduces a smearing effect on energy and momentum which is washed out after kinematical integrations. It should also be pointed out that viewing a short-lived object such as a cluster as an asymptotic state with definite total energy and momentum is certainly an approximation and a slight violation of energy-momentum conservation is not to be taken as a serious awkwardness. Problems may arise only in handling single cluster collision events, where final states must have the energy and momentum of the colliding system.

Equation (3) can be written as

$$\begin{aligned} \Gamma_f &= \langle f|W|i\rangle \langle i|W^\dagger|f\rangle = \langle f|WP_iW^\dagger|f\rangle \\ &= |\eta_i|^2 \langle f|P_V P_i P_V^\dagger|f\rangle, \end{aligned} \quad (5)$$

where P_i is the projector over the initial quantum state and η_i is such that $\hat{\eta}|i\rangle = \eta_i|i\rangle$. In principle, the projection is to be carried out onto a state with definite energy, momentum, spin (the Pauli-Lubanski vector), parity, C -parity (if the cluster is neutral) and internal charges. Hence the most general projector to be considered reads

$$P_i = P_{P,J,\lambda,\pi} P_\chi P_{I,I_3} P_{\mathbf{Q}}, \quad (6)$$

where P is the four-momentum of the cluster, J the spin, λ the helicity, π the parity, χ the C -parity, I and I_3 the

isospin and its third component and $\mathbf{Q} = (Q_1, \dots, Q_M)$ a set of M abelian (i.e. additive) charges such as baryon number, strangeness, electric charge etc. Of course, the projection P_χ makes sense only if $I_3 = 0$ and $\mathbf{Q} = \mathbf{0}$; in this case, P_χ commutes with all other projectors.

A state with definite four-momentum, spin, helicity and parity transforms according to an irreducible unitary representation ν of the extended Poincaré group $\text{IO}(1,3)^\dagger$, and the projector $P_{P,J,\lambda,\pi}$ can be written by using the invariant, suitably normalized, measure μ as

$$P_{P,J,\lambda,\pi} = \frac{1}{2} \sum_{z=1,\Pi} \dim \nu \int d\mu(g_z) D^{\nu\dagger}(g_z)_i^i U(g_z), \quad (7)$$

where z is the identity or space inversion Π , $g_z \in \text{IO}(1,3)^\dagger$, $D^\nu(g_z)$ is the matrix of the irreducible representation ν the initial state i belongs to, and $U(g_z)$ is the unitary representation of g_z in the Hilbert space. Similar integral expressions can be written for the projectors onto internal charges, for the groups $\text{SU}(2)$ (isospin) and $\text{U}(1)$ (for additive charges). Although projection operators cannot be rigorously defined for non-compact groups, such as the Poincaré group, we will maintain this naming, relaxing mathematical rigor. In fact, for non-compact groups, the projection operators cannot be properly normalized so as to have $P^2 = P$ and this is indeed related to the fact that $|i\rangle$ has infinite norm. Still, we will not be concerned with such drawbacks thereafter, whilst it will be favorable to keep the projector formalism. Working in the rest frame of the cluster, with $P = (M, \mathbf{0})$, the matrix element $D^{\nu\dagger}(g_z)_i^i$ vanishes, unless the Lorentz transformations are pure rotations, and this implies the reduction of the integration in (7) from $\text{IO}(1,3)^\dagger$ to the subgroup $\text{T}(4) \otimes \text{SU}(2) \otimes \mathbb{Z}_2$ (see Appendix B). Altogether, the projector $P_{P,J,\lambda,\pi}$ reduces to

$$\begin{aligned} P_{P,J,\lambda,\pi} &= \frac{1}{(2\pi)^4} \int d^4x e^{iP \cdot x} U(\text{T}(x)) \\ &\times (2J+1) \int dR D^J(R) \lambda^* U(R) \frac{1 + \pi U(\Pi)}{2}, \end{aligned} \quad (8)$$

dR being the invariant $\text{SU}(2)$ measure normalized to 1. The invariant measure d^4x of the translation subgroup has been normalized with a coefficient $1/(2\pi)^4$ in order to lead to a Dirac delta, as shown below. This is indeed the general expression of the projector defining the proper microcanonical ensemble, where all conservation laws related to space-time symmetries are fulfilled.

Hereafter, we will confine ourselves to clusters with fixed energy, momentum and abelian charges while conservation of angular momentum, isospin, parity and C -parity will be disregarded. This is expected to be a very good approximation in high energy collisions, where many clusters are formed and these latter constraints should not play a significant role [10, 15]. On the other hand, they cannot be disregarded in very small hadronizing systems (e.g. $p\bar{p}$ at rest [16]) and, in such circumstances, the full projection operation in (8) should be carried out. As has been mentioned in the Introduction, the set of states with fixed values of energy, momentum and abelian charges is defined

as a microcanonical ensemble as well, and we will stick to this convention.

Dealing with clusters with an unspecified value of angular momentum isospin and parity means, from a statistical mechanics point of view, that all possible projections over definite values of those quantum numbers occur with their statistical weight. In other words, we shall sum over all $J, \lambda, I, I_3, \pi, \chi$, which amounts to simply removing the relevant projection operators in virtue of the completeness relations such as, for instance,

$$\sum_{J\lambda} (2J+1) \int dR D^J(R) \lambda^* U(R) = 1. \quad (9)$$

In this case, the projector operator onto the initial state reduces to the more familiar form

$$\begin{aligned} P_i &\rightarrow P_P P_Q \\ &= \frac{1}{(2\pi)^4} \int d^4x e^{iP \cdot x} e^{-iP_{op} \cdot x} \\ &\quad \times \frac{1}{(2\pi)^M} \int_{-\pi}^{+\pi} d^M \phi e^{i\mathbf{Q} \cdot \phi} e^{-i\mathbf{Q}_{op} \cdot \phi} \\ &= \delta^4(P - P_{op}) \delta_{\mathbf{Q}, \mathbf{Q}_{op}}, \end{aligned} \quad (10)$$

where $\pi = (\pi, \dots, \pi)$ and the group generators P_{op} and \mathbf{Q}_{op} have been introduced. The appearance of Dirac and Kronecker deltas in (10) reflects the abelian nature of the leftover space-time translations and $U(1)$ groups. By using the latter expression of the projector P_i , (4), and inserting two identity resolutions, (5) turns into

$$\begin{aligned} \Gamma_f &= |\eta_i|^2 \sum_{h_V h'_V} \sum_{f' f''} \langle f | h_V \rangle \langle h_V | f' \rangle \langle f'' | h'_V \rangle \langle h'_V | f \rangle \\ &\quad \times \langle f' | \delta^4(P - P_{op}) \delta_{\mathbf{Q}, \mathbf{Q}_{op}} | f'' \rangle, \end{aligned} \quad (11)$$

and, taking f', f'' states as energy-momentum and charges eigenstates:

$$\begin{aligned} \Gamma_f &= \sum_{h_V h'_V} \sum_{f'} \langle f | h_V \rangle \langle h_V | f' \rangle \langle f' | h'_V \rangle \langle h'_V | f \rangle \\ &\quad \times \delta^4(P - P_{f'}) \delta_{\mathbf{Q}, \mathbf{Q}_{f'}}; \end{aligned} \quad (12)$$

that is,

$$\Gamma_f = |\eta_i|^2 \sum_{f'} \left| \sum_{h_V} \langle f | h_V \rangle \langle h_V | f' \rangle \right|^2 \delta^4(P - P_{f'}) \delta_{\mathbf{Q}, \mathbf{Q}_{f'}}. \quad (13)$$

Multiparticle states in the Fock space are characterized by a set of integer occupation numbers for all the species and for all the kinematical states. This also applies to the general state $|h_V\rangle$ as long as it represents, as it has been assumed, free hadron and resonance states within the cluster, so one can write $|h_V\rangle = |\{\tilde{N}_j\} k_V\rangle$, where $\{\tilde{N}_j\} = (\tilde{N}_1, \dots, \tilde{N}_K)$ is a K -tuple of integer numbers, one for each hadron species j , and k_V denotes a set of kinematical variables, depending on the spatial region with volume V , describing the state of the

$\tilde{N} = \tilde{N}_1 + \tilde{N}_2 + \dots + \tilde{N}_K$ particles. Similarly, we can rewrite the states belonging to the complete basis as $|f\rangle = |\{N_j\} k\rangle$, where now k is meant to be a set of proper momenta and polarizations. Note that the expression (13) allows transitions to occur to states $|f\rangle$ with energy-momentum different from P , unless the volume is infinitely large. This tells us, as has been mentioned, that the energy-momentum spread is of the order of the inverse of the cluster's linear size.

To further develop (13) we shall assume that

$$\langle \{\tilde{N}_j\} k_V | \{N_j\} k \rangle = 0 \quad \text{if } \tilde{N}_j \neq N_j \quad \forall j. \quad (14)$$

Hence, it is required that states with different particle compositions, either within the bounded region or in the whole space, are orthogonal. Indeed, there are two contraindications to this assumption. The first is of a more fundamental character: in relativistic quantum field theory a condition like (14) cannot be exactly true, as stationary states localized in a finite region are not eigenstates of the properly defined particle number operator (localization involves the creation of particle-antiparticle pairs). However, this effect is relevant if the size of the region is lower than the Compton wavelength of the particle $1/m$, which is at most (for pions) ≈ 1.4 fm, corresponding to a volume of ≈ 3 fm³; for all other hadrons, this volume is significantly smaller. Henceforth, we will assume that the volumes to be dealt with are larger (not too much though) and will take a non-relativistic quantum-mechanical treatment as a good approximation. The second one is concerned with strongly decaying resonances, which, in principle, should not be orthogonal to the states of their decay products; however, we have assumed that resonances are to be treated as independent states, so the orthogonality relation is correct in the framework of the ideal hadron-resonance gas.

With these two caveats in mind, we proceed to calculate the total rate of some *channel*, i.e. a multihadronic configuration $\{N_j\}$, by summing over the physical observables k , being $|f\rangle = |\{N_j\} k\rangle$. Applying the sum to the right hand side of (12), taking into account the condition (14) and the completeness of the set $|\{N_j\} k\rangle$, one obtains

$$\begin{aligned} &\sum_k \langle f | h_V \rangle \langle h'_V | f \rangle \\ &= \prod_j \delta_{N_j \tilde{N}_j} \delta_{N_j \tilde{N}'_j} \sum_k \langle \{N_j\} k | \{N_j\} k_V \rangle \langle \{N_j\} k'_V | \{N_j\} k \rangle \\ &= \langle h_V | h'_V \rangle \prod_j \delta_{N_j \tilde{N}_j} = \delta_{h_V h'_V} \prod_j \delta_{N_j \tilde{N}_j}, \end{aligned} \quad (15)$$

and, therefore

$$\begin{aligned} \Gamma_{\{N_j\}} &\equiv \sum_k \Gamma_{\{N_j\}, k} \\ &= |\eta_i|^2 \sum_{k'} \sum_{k_V} |\langle \{N_j\} k' | \{N_j\} k_V \rangle|^2 \delta^4(P - P_{f'}) \delta_{\mathbf{Q}, \mathbf{Q}_{f'}}. \end{aligned} \quad (16)$$

The above equation (and, maybe more apparently, (19) below) shows that only kinematical states fulfilling energy-momentum conservation contribute to the total rate of a

channel, even though the transition to final states with $P_f \neq P$ is allowed, as discussed.

We are now in a position to recover an expression like (1) mentioned at the beginning of this section. More specifically, we can prove that the right hand side in (16) is $|\eta_i|^2$ times the usual expression of the probability of the multihadronic configuration $\{N_j\}$ to occur in the microcanonical ensemble of an ideal hadron-resonance gas with four-momentum P , charges \mathbf{Q} and volume V , as long as the aforementioned relativistic quantum field effects are disregarded. We will show this first in the simple case of a channel with all different particles, i.e. $N_j \leq 1 \forall j$; the case of identical particles will be handled in the next section. The scalar product in (16) factorizes, so that

$$\sum_{k_V} |\langle \{N_j\} k' | \{N_j\} k_V \rangle|^2 = \prod_{i=1}^N \sum_{\mathbf{k}_i, \tau_i} |\langle \mathbf{p}'_i \sigma'_i | \mathbf{k}_i \tau_i \rangle|^2, \quad (17)$$

where $i = 1, \dots, N$ is the single-particle index. The variable \mathbf{p} is a momentum whilst \mathbf{k} denotes three variables defining the state of the particle within the region with volume V (e.g. a plane wave vector for a rectangular box or energy and angular momenta for a sphere). The variables σ'_i and τ_i label different polarization states of the particle and may refer to different projections of the spin (or the helicity); we will assume that the transformation from τ to σ is unitary. As long as $|\mathbf{k}, \tau\rangle$ is a complete set of one-particle states in the region with volume V , as a consequence of the completeness of the states $|h_V\rangle$, it can be shown that, in the non-relativistic quantum mechanics approximation (see Appendix C),

$$\sum_{\mathbf{k}, \tau} |\langle \mathbf{p} \sigma | \mathbf{k} \tau \rangle|^2 = \frac{V}{(2\pi)^3}. \quad (18)$$

Thus, taking into account that $\sum_{k'} = \left[\prod_{i=1}^N \sum_{\sigma_i} \int d^3 \mathbf{p}_i \right]^2$ and (17) and (16) become

$$\begin{aligned} \Gamma_{\{N_j\}} &= |\eta_i|^2 \frac{V^N}{(2\pi)^{3N}} \left[\prod_{i=1}^N (2J_i + 1) \int d^3 \mathbf{p}_i \right] \delta^4 \left(P - \sum_i p_i \right) \\ &= |\eta_i|^2 \Omega_{\{N_j\}}, \end{aligned} \quad (19)$$

where charge conservation, $\sum_j N_j \mathbf{q}_j = \mathbf{Q}$, is understood. Therefore, the rate $\Gamma_{\{N_j\}}$ is proportional to the usual expression of the *phase space volume* or density of states per four-momentum cell $\Omega_{\{N_j\}}$ of the multihadronic configuration $\{N_j\}$ in the microcanonical ensemble of the ideal hadron-resonance gas. It is to be emphasized that this formula, which has been used by many authors in the framework of the statistical model, is not the most general though, as all particles must be different. Therefore, it corresponds

² The notation $\left[\prod_i \int d^3 \mathbf{p}_i \right]$ stands for the integral operator $\int d^3 \mathbf{p}_1 \dots \int d^3 \mathbf{p}_N$ and it is understood to act on its right hand side argument

to assuming classical Boltzmann statistics. We will see in the next section that, if quantum statistics is taken into account, the integral in (19) is indeed a single term of an expansion.

Even though our derivation might look unnecessarily elaborated, the N -body relativistic phase space volume in Boltzmann statistics (19) has been recovered starting from purposely built quantum-mechanical transition probabilities without invoking any time-reversal argument or averaging procedures like in previous treatments [9, 10]. Furthermore, it should be emphasized that this derivation is more general than previous treatments because we did not consider particle states within the cluster as energy-momentum eigenstates. In fact, (19) is obtained in a traditional approach [11] by working out the expression

$$\Omega_{\{N_j\}} = \sum_{\text{states}} \delta^4(P - P_{\text{state}}) \delta_{\{N_j\}, \{N_j\}_{\text{state}}}, \quad (20)$$

with the key assumption that particles within the cluster have indeed definite four-momenta and so does the whole multiparticle state:

$$P_{\text{state}} = \sum_i p_i, \quad (21)$$

and finally replacing the sum over particle states in the cluster with an integration:

$$\sum_{\mathbf{k}} \rightarrow \frac{V}{(2\pi)^3} \int d^3 \mathbf{p}. \quad (22)$$

Altogether, this approach can be a good approximation only for very large volumes, because only for large volumes can the uncertainty on energy-momentum entailed by localization within the cluster be negligible. For smaller volumes, the localized multiparticle states, that we have denoted as $|h_V\rangle$, are not eigenstates of energy-momentum and their spread in energy-momentum cannot be neglected. It is worth making a rough estimate of how large the volume ought to be for the traditional approach to be valid. This can be done in two ways: requiring that the uncertainty in momentum for a single-particle localized state is not larger than of the order of, say, 10%, or arguing that the approximation (22) is indeed a good one provided that the number of phase space cells is at least of the order of 10–100. Working out (22) or using the indeterminacy principle, it turns out in both cases that the cluster's linear size should be larger than $\approx (6-10)/p$, where p is the typical momentum of the particles at hadronization. Since this is of the order of some hundreds MeV, the linear size must be of the order of, say, 3–10 fm, which is consistently larger than the limit set by the aforementioned condition on the Compton wavelength of the particles, of the order of a fraction of a fermi. Therefore, the requirement on the volume for the validity of the traditional treatment is more stringent than that needed for the present one.

Although in the case of Boltzmann statistics, the traditional and the present approach lead to the same expression (19) for the N -body relativistic phase space volume,

different expressions are found in the case of quantum statistics, that is, with identical particles, as will be shown in the next section.

3 Identical particles and cluster decomposition

If there are identical particles in the channel, (16) holds, but (17) does not and has to be modified. For the sake of simplicity, we will start with the case of only one kind of particle in the channel and assume that charge conservation is fulfilled. As has already been mentioned, relativistic quantum field effects will be disregarded, namely, the cluster's size is assumed to be significantly larger than the Compton wavelength of the particle. The correspondence between Fock space and multiparticle tensor space requires the identification

$$|\{N_j\}k_V\rangle \rightarrow \sum_p \chi(p)^b \frac{1}{\sqrt{N!n_1! \dots n_M!}} |\mathbf{k}_{p(1)}\tau_{p(1)}, \dots, \mathbf{k}_{p(N)}\tau_{p(N)}\rangle, \quad (23)$$

where p is a permutation of the integers $1, \dots, N$ and $\chi(p)$ its parity; the n_i are the number of times a given vector \mathbf{k}_i recurs in the state with $\sum_{i=1}^M n_i = N$; $b = 0$ for bosons and $b = 1$ for fermions. As there is only one particle species, the phase space volume $\Omega_{\{N_j\}}$ can be denoted with Ω_N and can be calculated by using (16). Replacing $|\{N_j\}k'$ with

$$|\{N_j\}k\rangle \rightarrow \sum_p \frac{\chi(p)^b}{\sqrt{N!}} |\mathbf{p}_{p(1)}\sigma_{p(1)}, \dots, \mathbf{p}_{p(N)}\sigma_{p(N)}\rangle \quad (24)$$

similarly to (23), and dividing by $1/N!$ in order to avoid multiple counting of (anti-) symmetric basis tensors when integrating over all possible momenta, we find

$$\begin{aligned} \Omega_N &= \frac{\Gamma_N}{|\eta_i|^2} \\ &= \left[\prod_{i=1}^N \sum_{\sigma_i} \int d^3\mathbf{p}_i \right] \delta^4(P - P_f) \\ &\quad \times \sum_{k_V} \left| \sum_p \chi(p)^b \frac{1}{\sqrt{N!n_1! \dots n_M!}} \right. \\ &\quad \left. \times \langle \mathbf{p}_1\sigma_1, \dots | \mathbf{k}_{p(1)}\tau_{p(1)}, \dots \rangle \right|^2. \end{aligned} \quad (25)$$

In (25) and hereafter P_f must be understood as the sum of the four-momenta of all particles in the channel. The last factor in the above equation can be worked out as follows:

$$\begin{aligned} &\sum_{k_V} \left| \sum_p \chi(p)^b \frac{1}{\sqrt{N!n_1! \dots n_M!}} \langle \mathbf{p}_1\sigma_1, \dots | \mathbf{k}_{p(1)}\tau_{p(1)}, \dots \rangle \right|^2 \\ &= \sum_{k_V} \frac{1}{N!n_1! \dots n_M!} \sum_{p,q} \chi(p)^b \chi(q)^b \end{aligned} \quad (26)$$

$$\begin{aligned} &\times \langle \mathbf{p}_1\sigma_1, \dots | \mathbf{k}_{p(1)}\tau_{p(1)}, \dots \rangle \langle \mathbf{k}_{q(1)}\tau_{q(1)}, \dots | \mathbf{p}_1\sigma_1, \dots \rangle \\ &= \frac{1}{N!^2} \sum_{p,q} \chi(pq)^b \prod_{i=1}^N \sum_{\mathbf{k}_i\tau_i} \langle \mathbf{p}_{p(i)}\sigma_{p(i)} | \mathbf{k}_i\tau_i \rangle \langle \mathbf{k}_i\tau_i | \mathbf{p}_{q(i)}\sigma_{q(i)} \rangle, \end{aligned}$$

where, in the last equality, we have redefined the dummy permutation indices p, q as their inverse and multiplied each term by a factor $n_1! \dots n_M! / N!$ in order to avoid multiple counting of the symmetric (antisymmetric) basis tensors $|h_V\rangle$ when the sum over all possible vectors \mathbf{k} and polarizations τ is carried out. Finally, taking into account that also \mathbf{k}_i and τ_i are dummy indices, one sum over permutations can be trivially performed and we are left with the transformation

$$\sum_k \sum_{k_V} |\langle \{N_j\}k | \{N_j\}k_V \rangle|^2 \rightarrow \quad (27)$$

$$\frac{1}{N!} \sum_r \chi(r)^b \left[\prod_{i=1}^N \int d^3\mathbf{p}_i \right] \sum_{\mathbf{k}_i\tau_i} \langle \mathbf{p}_i\sigma_i | \mathbf{k}_i\tau_i \rangle \langle \mathbf{k}_i\tau_i | \mathbf{p}_{r(i)}\sigma_{r(i)} \rangle,$$

with $r = p^{-1}q$ and $\chi(r) = \chi(p^{-1}q) = \chi(p^{-1})\chi(q) = \chi(p)\chi(q)$. The inner sums in the above equality yield (see Appendix C)

$$\begin{aligned} &\sum_{\mathbf{k}_i\tau_i} \langle \mathbf{p}_i\sigma_i | \mathbf{k}_i\tau_i \rangle \langle \mathbf{k}_i\tau_i | \mathbf{p}_{r(i)}\sigma_{r(i)} \rangle \\ &= \frac{\delta_{\sigma_r(i)}^{\sigma_i}}{(2\pi)^3} \int d^3\mathbf{x} e^{i\mathbf{x}\cdot(\mathbf{p}_{r(i)} - \mathbf{p}_i)}, \end{aligned} \quad (28)$$

so the following expression of the phase space volume Ω_N for N identical particles is obtained:

$$\begin{aligned} \Omega_N &= \sum_r \frac{\chi(r)^b}{N!} \sum_{\sigma_1, \dots, \sigma_N} \int d^3\mathbf{p}_1 \dots d^3\mathbf{p}_N \delta^4(P - P_f) \\ &\quad \times \prod_{i=1}^N \delta_{\sigma_r(i)}^{\sigma_i} \frac{1}{(2\pi)^3} \int d^3\mathbf{x} e^{i\mathbf{x}\cdot(\mathbf{p}_{r(i)} - \mathbf{p}_i)}. \end{aligned} \quad (29)$$

Hence, the phase space volume of N identical particles is given by the sum of $N!$ terms and it is thus enhanced or suppressed with respect to the case of distinguishable particles. As will be proved in the following, this effect is owing to the finite volume and, thereby, this model naturally accounts for Bose–Einstein and Fermi–Dirac correlations.

To develop (29), it is useful to recall that any permutation r of N integers can be uniquely decomposed into the product of cyclic permutations, that is $r = c_1 \dots c_H$. Let n be the number of integers in each cyclic permutation and let h_n be the number of cyclic permutations with n elements in r so that $\sum_{n=1}^{\infty} n h_n = N$. The set of integers $h_1, \dots, h_N \equiv \{h_n\}$, with $\sum_{n=1}^{\infty} h_n \equiv H$, is usually defined as a *partition* and different permutations having the same structure of cyclic decomposition, that is, the same number of integers for each c_l (i.e. the same partition), belong to the same conjugacy class of the permutation group S_N . The crucial observation is that each term in (29) is invariant over a conjugacy class, or, in other words, depends

only on the partition $\{h_n\}$; this happens because different permutations in the same conjugacy class differ only by a redefinition of the integers $1, \dots, N$ and this is just a change of the name of the dummy integration variables and sum indices in (29). The number of permutations of S_N belonging to a given conjugacy class is a well known number [17], namely $N! / \prod_{n=1}^N n^{h_n} h_n!$. Furthermore, for the representative cyclic permutation c with n elements $(1, \dots, n)$:

$$\sum_{\sigma_1, \dots, \sigma_n} \prod_{i=1}^n \delta_{\sigma_{c(i)}}^{\sigma_i} = (2J+1), \quad (30)$$

so that

$$\sum_{\sigma_1, \dots, \sigma_N} \prod_{i=1}^N \delta_{\sigma_{r(i)}}^{\sigma_i} = \prod_{l=1}^H \sum_{\sigma_1, \dots, \sigma_{n_l}} \prod_{i=1}^{n_l} \delta_{\sigma_{c_l(i_l)}}^{\sigma_i} = (2J+1)^H, \quad (31)$$

where $r = c_1 \dots c_H$ and n_l is the number of integers in the cyclic permutation c_l . By defining

$$F_{n_l} = \prod_{i=1}^{n_l} \frac{1}{(2\pi)^3} \int_V d^3 \mathbf{x} e^{i\mathbf{x} \cdot (\mathbf{p}_{c_l(i_l)} - \mathbf{p}_{i_l})}, \quad (32)$$

and taking into account that $\chi(c_l) = (-1)^{n_l+1}$, we can finally rewrite (29) as

$$\begin{aligned} \Omega_N &= \sum_{\{h_n\}} \Omega_N(\{h_n\}) \\ &= \sum_{\{h_n\}} \frac{(\mp 1)^{N+H} (2J+1)^H}{\prod_{n=1}^N n^{h_n} h_n!} \left[\prod_{i=1}^N \int d^3 \mathbf{p}_i \right] \\ &\quad \times \delta^4(P - P_f) \prod_{l=1}^H F_{n_l}, \end{aligned} \quad (33)$$

where the upper sign applies to fermions, the lower to bosons. Therefore, the phase space volume for a channel with N identical particles consists of a large number of terms $\Omega_N(\{h_n\})$, each corresponding to a partition $\{h_n\}$, which is usually called in statistical mechanics a *cluster decomposition*³.

In the large volume limit, the dominant term is the one with the highest power of V and this corresponds to the partition $(h_1, h_2, \dots, h_N) = (N, 0, \dots, 0)$, i.e. the identical permutation. In this case there are N factors $F_1 = V/(2\pi)^3$ and the whole term reads

$$\begin{aligned} \Omega_N(N, 0, \dots, 0) & \\ &= \left[\frac{V(2J+1)}{(2\pi)^3} \right]^N \frac{1}{N!} \int d^3 \mathbf{p}_1 \dots d^3 \mathbf{p}_N \delta^4(P - P_f), \end{aligned} \quad (34)$$

which is the phase space volume for a set of N identical particles in classical Boltzmann statistics; we have indeed recovered the phase space volume quoted in (19). All other

³ The term cluster in this context has nothing to do with our previous definition of an individual hadronizing source

terms of the expansion in (33) have a lower power of V . The next-to-leading term corresponds to the conjugacy class of permutations with one exchange and $N-2$ unchanged integers, i.e. $(h_1, h_2, h_3, \dots, h_N) = (N-2, 1, 0, \dots, 0)$. In the large volume limit it is easily seen, looking at (32), that $F_2 \rightarrow \delta^3(\mathbf{p}_2 - \mathbf{p}_1) V / (2\pi)^3$ and the whole term thus reads

$$\begin{aligned} \Omega_N(N-2, 1, 0, \dots, 0) & \\ &= \frac{(\mp 1)}{2(N-2)!} \left[\frac{V(2J+1)}{(2\pi)^3} \right]^{N-1} \\ &\quad \times \int d^3 \mathbf{p}_2 \dots d^3 \mathbf{p}_N \delta^4(P - P_f), \end{aligned} \quad (35)$$

where $P_f = 2p_2 + p_3 + \dots p_N$. Introducing the new integration variables $\mathbf{p}' = 2\mathbf{p}_2$ the energy term $2\varepsilon_2$ becomes $\sqrt{\mathbf{p}'^2 + (2m)^2}$ and (35) can be rewritten as

$$\begin{aligned} \Omega_N(N-2, 1, 0, \dots, 0) & \\ &= \frac{(\mp 1)}{2^4(N-2)!} \left[\frac{V(2J+1)}{(2\pi)^3} \right]^{N-1} \\ &\quad \times \int d^3 \mathbf{p}' d^3 \mathbf{p}_3 \dots d^3 \mathbf{p}_N \delta^4 \left(P - p' - \sum_{i=3}^N p_i \right). \end{aligned} \quad (36)$$

Aside from the sign and an overall normalization factor $1/16$, this term corresponds to the Boltzmann limit (35) of the phase space volume for a set of $N-2$ identical particles plus a new particle (labelled with a prime) obtained by clumping particles 1 and 2 into a lump with a mass twice the mass of 1 and 2 and the same spin.

Actually, this kind of interpretation holds for all of the terms in (33). In fact, in the large volume limit, each F_n implies the elimination of $n-1$ integration variables through the appearance of Dirac deltas, while a single $V/(2\pi)^3$ factor is left because of the cyclic structure of the permutation, namely

$$F_{n_l} \rightarrow \frac{V}{(2\pi)^3} \prod_{i=1}^{n_l-1} \delta^3(\mathbf{p}_{i_l} - \mathbf{p}_{i_l+1}). \quad (37)$$

Then, after trivial integrations are carried out in (33), the Dirac delta forcing conservation of four-momentum turns into $\delta^4(P_i - n_1 p_1 - n_2 p_{n_1+1} \dots - n_H p_{n_{H-1}+1})$ and new integrations variables can be introduced:

$$\mathbf{p}'_1 = n_1 \mathbf{p}_1 \quad \mathbf{p}'_2 = n_2 \mathbf{p}_{n_1+1} \quad \dots \quad \mathbf{p}'_H = n_H \mathbf{p}_{n_{H-1}+1}, \quad (38)$$

as well as new energies:

$$\varepsilon' = n_l \varepsilon = n_l \sqrt{\mathbf{p}'^2 + m^2} = \sqrt{\mathbf{p}'^2 + (n_l m)^2}. \quad (39)$$

Therefore, the term corresponding to the partition $\{h_n\}$ can be written as

$$\begin{aligned} \Omega_N(\{h_n\}) &= \frac{(\mp 1)^{N+H}}{\prod_{n=1}^N n^{h_n} h_n! \prod_{l=1}^H n_l^3} \left[\frac{V(2J+1)}{(2\pi)^3} \right]^H \\ &\quad \times \int d^3 \mathbf{p}'_1 \dots d^3 \mathbf{p}'_H \delta^4 \left(P - \sum_{l=1}^H p'_l \right), \end{aligned} \quad (40)$$

where particles are now clumped into H lumps with mass equal to $n_l m$ and spin J . Since $\prod_{l=1}^H n_l^3 = \prod_{n=1}^N n^{3h_n}$, the above equation can also be written as

$$\Omega_N(\{h_{n_j}\}) = \frac{(\mp 1)^{N+H}}{\prod_{n=1}^N n^{4h_n} h_n!} \left[\frac{V(2J+1)}{(2\pi)^3} \right]^H \quad (41)$$

$$\times \int d^3 \mathbf{p}'_1 \dots d^3 \mathbf{p}'_H \delta^4 \left(P - \sum_{l=1}^H p'_l \right).$$

We can thus conclude that the general term relevant to the cluster decomposition of the phase space volume of a set of N identical particles can be obtained by calculating the phase space volume, in Boltzmann statistics, of a suitable set of lumps having as mass multiple integer values of m and spin J , weighted by an overall coefficient of $(\mp 1)^{N-H} / \prod_n n^{4h_n}$. Note that the factors $1/h_n!$ already take into account the identity of the lumps.

After having inferred the expressions of the phase space volume of a channel with N identical particles, the generalization to a channel $\{N_j\}$ (see Sect. 2) with an arbitrary number of groups of identical particles for each species j is rather straightforward and can be achieved by going along the previous arguments. Thereby, the following equations are obtained which are extensions of (27), (29), and (33), respectively:

$$\sum_k \sum_{k_V} |\langle \{N_j\} k | \{N_j\} k_V \rangle|^2 \rightarrow \prod_j \frac{1}{N_j!} \sum_{r_j \in S_{N_j}} \chi(r_j)^{b_j} \quad (42)$$

$$\times \left[\prod_{i_j=1}^{N_j} d^3 \mathbf{p}_{i_j} \right]_{\mathbf{k}_{i_j} \tau_{i_j}} \sum \langle \mathbf{p}_{i_j} \sigma_{i_j} | \mathbf{k}_{i_j} \tau_{i_j} \rangle \langle \mathbf{k}_{i_j} \tau_{i_j} | \mathbf{p}_{r_j(i_j)} \sigma_{r_j(i_j)} \rangle,$$

$$\Omega_{\{N_j\}} = \sum_{\sigma_1, \dots, \sigma_N} \int d^3 \mathbf{p}_1 \dots d^3 \mathbf{p}_N \delta^4(P - P_f) \quad (43)$$

$$\times \prod_j \sum_{r_j \in S_{N_j}} \frac{\chi(r_j)^{b_j}}{N_j!} \prod_{i_j=1}^{N_j} \frac{\delta_{\sigma_{r_j(i_j)}}^{\sigma_{i_j}}}{(2\pi)^3} \int_V d^3 \mathbf{x} e^{i\mathbf{x} \cdot (\mathbf{p}_{r_j(i_j)} - \mathbf{p}_{i_j})}$$

and

$$\Omega_{\{N_j\}} = \int d^3 \mathbf{p}_1 \dots d^3 \mathbf{p}_N \delta^4(P - P_f) \quad (44)$$

$$\times \prod_j \sum_{\{h_{n_j}\}} \frac{(\mp 1)^{N_j+H_j} (2J+1)^{H_j}}{\prod_{n_j=1}^{N_j} n_j^{h_{n_j}} h_{n_j}!} \prod_{l_j=1}^{H_j} F_{n_{l_j}},$$

with $H_j = \sum_{n_j=1}^{N_j} h_{n_j}$ and $N_j = \sum_{n_j=1}^{N_j} n_j h_{n_j}$. The above expression is the most general for the microcanonical phase space volume of the multihadronic channel $\{N_j\}$ in the ideal hadron-resonance gas framework with full quantum statistics and generalizes the expression obtained in [11]:

$$\Omega_{\{N_j\}} = \left[\prod_j \sum_{\{h_{n_j}\}} (\mp 1)^{N_j+H_j} \frac{1}{\prod_{n_j=1}^{N_j} n_j^{4h_{n_j}} h_{n_j}!} \right] \quad (45)$$

$$\times \left[\prod_{l_j=1}^{H_j} \frac{V(2J_j+1)}{(2\pi)^3} \int d^3 \mathbf{p}'_{l_j} \right] \delta^4 \left(P_i - \sum_{j,l_j=1}^{H_j} p'_{l_j} \right),$$

where, for a set of partitions $\{h_{n_1}\}, \dots, \{h_{n_K}\}$, the four-momenta p'_{l_j} are those of lumps of particles of the same species j (H_j in number) with mass $n_j m_j$ and spin J_j . While (44) is indeed the correct expression of the microcanonical relativistic phase space volume of the channel $\{N_j\}$ in the statistical hadronization model, (45) turns out to be a special case of (44) and in fact can be derived from it by replacing F_{n_l} with their limiting expressions (37). Thus, the expression (45) is a good approximation of (44) only in the limit of large volumes. In fact, the derivation of the relativistic phase space volume in (45) in [11] was based on the assumption that particle states within the cluster are energy-momentum eigenstates, which is a good approximation only for large volumes, as extensively discussed at the end of the previous section. For the expression (44) to be valid, we now just need that the linear size of the cluster must be sufficiently larger than the Compton wavelength of the involved particles in order to neglect relativistic quantum field effects.

The fact that the leading Boltzmann terms in the general expression (44) and the approximate one (45) are the same, as already pointed out in Sect. 2, reduces the actual numerical impact of this generalization on many, yet not all, observables. For instance, at the actual temperature values of about 160 MeV found in previous analyses of many high energy collisions in the canonical ensemble [1, 7], quantum statistics corrections on average particle multiplicities turned out to be significant for pions only (about 10%), whilst they can be neglected for all other hadrons. Therefore, as long as average multiplicities are concerned, the calculation can be done within Boltzmann statistics and the difference between the correct formula and the approximate one is almost irrelevant. Thus, the only effective requirement on the cluster size for the validity of all performed analyses in high energy collisions [1, 7] is that it must be larger than the Compton wavelength of the particles (at most 1.4 fm) and this is always met.

Even though average multiplicities are essentially unaffected in most practical cases, there are other observables which are sensitive to quantum statistics effects and for which the fully correct calculation of the phase space volume (45) is compelling, e.g. Bose-Einstein correlation spectra and multi-pion exclusive channel rates.

4 Microcanonical partition function

The overall phase space volume of the ideal hadron-resonance gas is obtained by summing $\Omega_{\{N_j\}}$ over all allowed channels:

$$\Omega = \sum_{\{N_j\}} \Omega_{\{N_j\}} \delta_{\mathbf{Q}, \mathbf{Q}_{\{N_j\}}}. \quad (46)$$

As $\Omega_{\{N_j\}} = \Gamma_{\{N_j\}} / |\eta_i|^2$, Ω can be expressed on the basis of (11) after having removed the two identity resolutions

in f' and f'' :

$$\begin{aligned}\Omega &= \frac{1}{|\eta_i|^2} \sum_f \Gamma_f \\ &= \sum_f \sum_{h_V, h'_V} \langle f | h_V \rangle \langle h_V | \delta^4(P - P_{\text{op}}) \delta_{\mathbf{Q}, \mathbf{Q}_{\text{op}}} | h'_V \rangle \langle h'_V | f \rangle \\ &= \sum_{h_V} \langle h_V | \delta^4(P - P_{\text{op}}) \delta_{\mathbf{Q}, \mathbf{Q}_{\text{op}}} | h_V \rangle.\end{aligned}\quad (47)$$

The last expression makes it apparent that the definition of Ω as the *microcanonical partition function* is an appropriate one. If the sums in (46) and (47) are performed over all channels regardless of their charge, the obtained quantity is defined as a *grand-microcanonical partition function*:

$$\Omega = \sum_{\{N_j\}} \Omega_{\{N_j\}} = \sum_{h_V} \langle h_V | \delta^4(P - P_{\text{op}}) | h_V \rangle. \quad (48)$$

Throughout this section we will confine ourselves to the latter, rather than to the properly defined microcanonical partition function, in order not to bring along a cumbersome formalism. This limitation shall not affect the generality of the expounded arguments, and the extension to the case of constrained charges is indeed straightforward.

We have seen in the previous section that (44) is a correct generalization of (45) for finite volumes. Likewise, (48) is a generalization of the expression quoted in the previous literature [7, 11]:

$$\Omega = \sum_{\text{states}} \delta^4(P - P_{\text{state}}). \quad (49)$$

In fact, (49) is a straightforward consequence of (48) if $|h_V\rangle$ is an eigenstate of energy-momentum. However, we have already emphasized that $|h_V\rangle$ is a localized state and its four-momentum is a well defined quantity only in the large volume limit, as has been discussed at the end of Sect. 2. Thus, (49) is consistent only if the cluster is sufficiently large, whereas (48) is always well defined. On the other hand, a closed analytical integral expression for the correct (grand-) microcanonical partition function cannot be written. The best one can do is to decompose it as a sum over channels, as in (46), and calculate numerically the $\Omega_{\{N_j\}}$'s according to (44), which is a formidable task indeed. Conversely, (49) does lead to a closed integral expression, which can be obtained by firstly expanding the Dirac delta in (49) as a Fourier integral:

$$\delta^4(P - P_{\text{state}}) = \frac{1}{(2\pi)^4} \int d^4 y e^{i(P - P_{\text{state}}) \cdot y}, \quad (50)$$

and reexpressing (49) as

$$\Omega = \frac{1}{(2\pi)^4} \int d^4 y e^{iP \cdot y} \sum_{\{n_{jh}\}} \prod_{j,h} e^{-in_{jh} p_{jh} \cdot y}. \quad (51)$$

The sum over states is in fact a sum over all possible occupation numbers of each phase space cell. The calculation now

proceeds by taking advantage of the commutability between sum and product in (51). However, unlike for fermions for which $n_{jh} = 0, 1$ only, the sum over occupation numbers does not converge to a finite value for bosons as n_{jh} runs from 0 to ∞ . The convergence is recovered if the time component of y is provided with a small negative imaginary part $-i\varepsilon$. If we introduce such a term in (51) the sums can be performed, and the result is

$$\begin{aligned}\Omega &= \lim_{\varepsilon \rightarrow 0} \frac{1}{(2\pi)^4} \int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} dy^0 \int d^3 y e^{iP \cdot y} \\ &\quad \times \exp \left[\sum_{j,h} \log(1 \pm e^{-ip_{jh} \cdot y})^{\pm 1} \right],\end{aligned}\quad (52)$$

where the upper sign applies to fermions, the lower to bosons. The integrand function is in fact singular for $y = 0$ and the shift of the integration contour in the complex plane provides a regularization prescription. The sum over phase space cells can be replaced, in the large volume limit, by an integration according to (22), so that Ω reads

$$\begin{aligned}\Omega &= \lim_{\varepsilon \rightarrow 0} \frac{1}{(2\pi)^4} \int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} dy^0 \int d^3 y e^{iP \cdot y} \\ &\quad \times \exp \left[\sum_j \frac{(2J_j + 1)V}{(2\pi)^3} \int d^3 \mathbf{p} \log(1 \pm e^{-ip \cdot y})^{\pm 1} \right].\end{aligned}\quad (53)$$

We will prove in the remainder of this section that the closed expression for the grand-microcanonical partition function, (53), can be recovered without invoking (49) and (22), but starting from the general expression (44) in at least two cases:

- (1) for Boltzmann statistics;
- (2) in a full quantum statistics treatment, by enforcing the approximation (37), namely

$$\frac{1}{(2\pi)^3} \int_V d^3 \mathbf{x} e^{i(\mathbf{p} - \mathbf{p}') \cdot \mathbf{x}} \simeq \delta^3(\mathbf{p} - \mathbf{p}'). \quad (54)$$

Henceforth, we will adopt the following shorthand:

$$\int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} d^4 y = \int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} dy^0 \int d^3 y. \quad (55)$$

- (1) Let us start by showing our case for Boltzmann statistics. We have seen in the previous section that confining ourselves to classical statistics amounts to retaining only the first term $\{h_{n_j}\} = (N_j, 0, \dots, 0)$ in the general cluster decomposition (44), hence $h_1 = H_j = N_j$, $F_1 = V/(2\pi)^3$ and

$$\begin{aligned}\Omega_{\{N_j\}}^{\text{Boltz}} &= \int d^3 \mathbf{p}_1 \dots d^3 \mathbf{p}_N \delta^4 \left(P - \sum_{i=1}^N p_i \right) \\ &\quad \times \prod_j \frac{1}{N_j!} \left[\frac{V(2J+1)}{(2\pi)^3} \right]^{N_j}.\end{aligned}\quad (56)$$

The Dirac delta in the above equation can be Fourier expanded; thus, after regularization

$$\begin{aligned} \Omega_{\{N_j\}}^{\text{Boltz}} &= \frac{1}{(2\pi)^4} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d^4y e^{iP \cdot y} \int d^3\mathbf{p}_1 \dots d^3\mathbf{p}_N \quad (57) \\ &\times \exp \left[-i \sum_{i=1}^N p_i \cdot y \right] \prod_j \frac{1}{N_j!} \left[\frac{V(2J+1)}{(2\pi)^3} \right]^{N_j} \\ &= \frac{1}{(2\pi)^4} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d^4y e^{iP \cdot y} \\ &\times \prod_j \frac{1}{N_j!} \left[\frac{V(2J+1)}{(2\pi)^3} \int d^3\mathbf{p} e^{-ip_j \cdot y} \right]^{N_j}. \end{aligned}$$

Summing over all channels yields the grand-microcanonical partition function:

$$\begin{aligned} \Omega^{\text{Boltz}} &= \frac{1}{(2\pi)^4} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d^4y e^{iP \cdot y} \\ &\times \exp \left[\sum_j \frac{V(2J+1)}{(2\pi)^3} \int d^3\mathbf{p} e^{-ip_j \cdot y} \right], \quad (58) \end{aligned}$$

which can be obtained indeed from (53) with the boltzmannian approximation:

$$\log(1 \pm e^{-ip \cdot y})^{\pm 1} \simeq e^{-ip \cdot y}. \quad (59)$$

This proves the first part of our argument.

(2) If quantum statistics is included, we make the supplementary assumption, as has been mentioned, that the approximations (37) apply and, thus, (44) turns to (45). Let us first restore $p_{l_j} = p'_{l_j}/n_{l_j}$ [see (38)] as integration variables and rewrite (45) by plugging in the Fourier expansion of the Dirac delta:

$$\begin{aligned} \Omega_{\{N_j\}} &= \left[\prod_j \sum_{\{h_{n_j}\}} (\mp 1)^{N_j+H_j} \frac{1}{\prod_{n_j=1}^{N_j} n_j^{h_{n_j}} h_{n_j}!} \right. \\ &\times \prod_{l_j=1}^{H_j} \left[\frac{V(2J+1)}{(2\pi)^3} \int d^3\mathbf{p}_{l_j} \right] \delta^4 \left(P_i - \sum_j \sum_{l_j=1}^{H_j} n_{l_j} p_{l_j} \right) \\ &= \frac{1}{(2\pi)^4} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d^4y e^{iP \cdot y} \left[\prod_j \sum_{\{h_{n_j}\}} \frac{(\mp 1)^{N_j+H_j}}{\prod_{n_j=1}^{N_j} n_j^{h_{n_j}} h_{n_j}!} \right. \\ &\times \prod_{l_j=1}^{H_j} \left[\frac{V(2J+1)}{(2\pi)^3} \int d^3\mathbf{p}_{l_j} \right] \exp \left[-i \sum_j \sum_{l_j=1}^{H_j} n_{l_j} p_{l_j} \cdot y \right] \\ &= \frac{1}{(2\pi)^4} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d^4y e^{iP \cdot y} \prod_j \sum_{\{h_{n_j}\}} \frac{(\mp 1)^{N_j+H_j}}{\prod_{n_j=1}^{N_j} n_j^{h_{n_j}} h_{n_j}!} \\ &\times \prod_{l_j=1}^{H_j} \left[\frac{V(2J+1)}{(2\pi)^3} \int d^3\mathbf{p}_{l_j} e^{-im_{l_j} p_{l_j} \cdot y} \right]. \quad (60) \end{aligned}$$

In order to simplify the notation, we introduce the quantities

$$z_j(n) \equiv z_j(n)(y) = \frac{V(2J+1)}{(2\pi)^3} \int d^3\mathbf{p} e^{-in p \cdot y}, \quad (61)$$

so that (60) can be further written as

$$\begin{aligned} \Omega_{\{N_j\}} &= \frac{1}{(2\pi)^4} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d^4y e^{iP \cdot y} \\ &\times \prod_j \sum_{\{h_{n_j}\}} (\mp 1)^{N_j+H_j} \frac{\prod_{l_j=1}^{H_j} z_j(n_{l_j})}{\prod_{n_j=1}^{N_j} n_j^{h_{n_j}} h_{n_j}!} \quad (62) \\ &= \frac{1}{(2\pi)^4} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d^4y e^{iP \cdot y} \prod_j \sum_{\{h_{n_j}\}} \prod_{n_j=1}^{\infty} \frac{(\mp 1)^{(n_j+1)h_{n_j}} z_j^{h_{n_j}}}{n_j^{h_{n_j}} h_{n_j}!}, \end{aligned}$$

where, in the last passage, we have taken advantage of the fact that $z_j(n)$ is constant over a conjugacy class. Also note that we have released the upper limit in the sum because of the constraint $\sum_{n_j} n_j h_{n_j} = N_j$ which effectively sets it to N_j . At this stage, the key observation is that we can implement this constraint through an integration in the complex plane for each species j and then perform an unconstrained sum over all h_{n_j} :

$$\sum_{\{h_{n_j}\}} = \sum_{h_1, \dots, h_\infty} \delta_{\sum_{n_j} n_j h_{n_j}, N_j} = \frac{1}{2\pi i} \oint \frac{dw}{w^{N_j+1}} \prod_{n_j=1}^{\infty} \sum_{h_{n_j}=0}^{\infty} w^{n_j h_{n_j}}, \quad (63)$$

so that the part of integrand in (62) following the j -product sign can be written as

$$\begin{aligned} &\frac{1}{2\pi i} \oint \frac{dw_j}{w_j^{N_j+1}} \prod_{n_j=1}^{\infty} \sum_{h_{n_j}=0}^{\infty} \frac{(\mp 1)^{(n_j+1)h_{n_j}} z_j^{h_{n_j}} w_j^{n_j h_{n_j}}}{n_j^{h_{n_j}} h_{n_j}!} \\ &= \frac{1}{2\pi i} \oint \frac{dw_j}{w_j^{N_j+1}} \prod_{n_j=1}^{\infty} \exp \left[\frac{(\mp 1)^{n_j+1}}{n_j} z_j(n_j) w_j^{n_j} \right] \\ &= \frac{1}{N_j!} \frac{d^{N_j}}{dw_j^{N_j}} \exp \left[\sum_{n_j=1}^{\infty} \frac{(\mp 1)^{n_j+1}}{n_j} z_j(n_j) w_j^{n_j} \right] \Bigg|_{w_j=0} \quad (64) \end{aligned}$$

By using the explicit expression of $z_j(n)$ in (61), the series in the exponential of (64) can be summed, and this yields, for the phase space volume $\Omega_{\{N_j\}}$,

$$\begin{aligned} \Omega_{\{N_j\}} &= \frac{1}{(2\pi)^4} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d^4y e^{iP \cdot y} \\ &\times \prod_j \frac{1}{N_j!} \frac{d^{N_j}}{dw_j^{N_j}} \exp \left[\frac{(2J+1)V}{(2\pi)^3} \int d^3\mathbf{p} \right. \\ &\times \log(1 \pm w_j e^{-ip \cdot y})^{\pm 1} \Bigg|_{w_j=0}. \quad (65) \end{aligned}$$

We are now in a position to calculate Ω by summing over all $N_j \forall j$, according to (48). The sum over each $N_j =$

$0, \dots, \infty$ can be performed independently and, noticing that each term is the N_j th one of the Taylor expansion of the exponential function evaluated at $w_j = 1$, one obtains

$$\Omega = \frac{1}{(2\pi)^4} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d^4 y e^{iP \cdot y} \quad (66)$$

$$\times \prod_j \exp \left[\frac{(2J_j + 1)V}{(2\pi)^3} \int d^3 \mathbf{p} \log(1 \pm e^{-ip \cdot y})^{\pm 1} \right],$$

which coincides with (53); this proves our second statement.

The recovery of the known expression of the microcanonical partition function in the two considered cases is not surprising as the same holds for the single channel phase space volume $\Omega_{\{N_j\}}$. We have seen this in Sect. 2 where it has been emphasized that $\Omega_{\{N_j\}}$ in Boltzmann statistics does not differ from its approximation in the large volume limit; and in Sect. 3, where we have seen that the $\Omega_{\{N_j\}}$ in full quantum statistics (45), deduced from (44) by enforcing the approximation (37), was obtained in the traditional approach [11] using (20), (21) and (22).

5 From microcanonical to canonical ensemble

What has been done for the grand-microcanonical ensemble can be straightforwardly extended to the properly called microcanonical ensemble by adding the further constraint of M abelian charges conservation, like in (46). The Kronecker delta can be Fourier expanded :

$$\delta_{\mathbf{Q}, \mathbf{Q}_{\{N_j\}}} = \prod_{m=1}^M \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi_m e^{i(Q_m - Q_{\{N_j\}m})\phi_m}$$

$$= \frac{1}{(2\pi)^M} \int_{-\pi}^{+\pi} d^M \phi e^{i(\mathbf{Q} - \mathbf{Q}_{\{N_j\}}) \cdot \phi}, \quad (67)$$

where the vector notation $\phi = (\phi_1, \dots, \phi_M)$ has been introduced. The reasoning in the previous section, from (56) onwards, can easily be repeated with the additional charge constraint (67), under the same conditions for the validity of the needed approximation (54). One can thus arrive at the following expression of the microcanonical partition function:

$$\Omega = \frac{1}{(2\pi)^{4+M}} \int d^4 y e^{iP \cdot y} \int_{-\pi}^{+\pi} d^M \phi e^{i\mathbf{Q} \cdot \phi} \quad (68)$$

$$\times \exp \left[\sum_j \frac{(2J_j + 1)V}{(2\pi)^3} \int d^3 \mathbf{p} \log(1 \pm e^{-ip \cdot y - i\mathbf{q}_j \cdot \phi})^{\pm 1} \right],$$

where $\mathbf{q}_j = (q_{j1}, \dots, q_{jM})$ are the abelian charges of the j th hadron species. Let us perform a rotation in the four-dimensional complex hyperplane by setting $z = iy$ and rewrite (68) as

$$\Omega = \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi i)^4} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d^4 z \exp[P \cdot z + \log Z(z, \mathbf{Q})], \quad (69)$$

where

$$Z(z, \mathbf{Q}) = \frac{1}{(2\pi)^M} \int_{-\pi}^{+\pi} d^M \phi e^{i\mathbf{Q} \cdot \phi} \quad (70)$$

$$\times \exp \left[\sum_j \frac{(2J_j + 1)V}{(2\pi)^3} \int d^3 \mathbf{p} \log(1 \pm e^{-z \cdot p - i\mathbf{q}_j \cdot \phi})^{\pm 1} \right].$$

In (70) the expression of the canonical partition function [1, 7] calculated for a *complex four-temperature* z is recognizable. The same expression can be obtained starting from the definition

$$Z(z, \mathbf{Q}) = \sum_{h_V} \langle h_V | e^{-z \cdot P_{\text{op}}} \delta_{\mathbf{Q}, \mathbf{Q}_{\text{op}}} | h_V \rangle, \quad (71)$$

and proceeding in the very same way as for the microcanonical partition function. Particularly, the approximations (54) are needed to get to (70).

If the volume and the mass of the cluster are large, one can make an approximate calculation of the integral in (69) through the saddle-point expansion. The large-valued parameter can be either the volume or the mass provided that the density M/V is a finite value, which is indeed the case of interest in the framework of the statistical hadronization model. The saddle-point four-vector β is determined by enforcing the vanishing of the integrand logarithmic derivative for each component μ :

$$\frac{\partial}{\partial z^\mu} [P \cdot z + \log Z(z, \mathbf{Q})] \Big|_{z=\beta}$$

$$= P_\mu + \frac{\partial}{\partial \beta^\mu} \log Z(\beta, \mathbf{Q}) = 0. \quad (72)$$

We assume that the above equation has one real solution (note that $Z(z)$ is real for a real argument, see (71)). This must be a timelike four-vector for integration in (70) to converge. Therefore, we can set $\beta = (1/T)\hat{u}$ where \hat{u} is a unit timelike vector and $T > 0$ is defined as a temperature, while β is usually called the temperature four-vector. It is not difficult to verify that if the cluster's rest frame is chosen, where $P = (M, \mathbf{0})$, β has vanishing spatial components and the usual expression of the canonical partition function is recovered:

$$Z(\mathbf{Q}) = \sum_{h_V} \langle h_V | e^{-H_{\text{op}}/T} \delta_{\mathbf{Q}, \mathbf{Q}_{\text{op}}} | h_V \rangle, \quad (73)$$

where H_{op} is the hamiltonian. Retaining only the leading term of the asymptotic expansion, the microcanonical partition function can be approximated as

$$\Omega \simeq \exp[P \cdot \beta + \log Z(\beta, \mathbf{Q})] \sqrt{\frac{1}{(2\pi)^4 \det \mathbf{H}(\beta, \mathbf{Q})}}, \quad (74)$$

where \mathbf{H} is the Hessian matrix $\partial^2 \log Z / \partial z^\mu \partial z^\nu$. In the cluster's rest frame $\beta = (\bar{\beta}, \mathbf{0})$ as already pointed out; thus, according to (70), the derivative $\partial \log Z / \partial z^i$ with respect to the spatial components of z vanishes because

of the odd-symmetric momentum integrands and, consequently, the Hessian determinant in (74) simply becomes $\partial^2 \log Z / \partial \bar{\beta}^2 = C_V T^2$. Altogether, if V is large, the microcanonical partition function Ω is proportional to the canonical partition function Z and we can write

$$\Omega(P, \mathbf{Q}) \stackrel{V \rightarrow \infty}{\propto} e^{\beta \cdot P} Z(\beta, \mathbf{Q}), \quad (75)$$

with β given by (72), and

$$Z(\beta, \mathbf{Q}) = \int d^4 P \theta(P^0) e^{-\beta \cdot P} \Omega(P, \mathbf{Q}). \quad (76)$$

This equation is indeed an exact one, as can be realized from (68); the canonical partition function is in fact the Laplace transform of the microcanonical one.

The question arises whether and in which range of values of the cluster's volume and mass the approximation (75), i.e. the use of the canonical ensemble, employed in several analyses of multiplicities in elementary collisions, is a good one for the calculation of relevant physical quantities. This issue can be tackled only numerically for the particular system of the ideal hadron-resonance gas, comparing the exact one with the approximate calculation; as has been mentioned in the Introduction, this will be the main subject of the second paper [12].

The way temperature has been introduced starting from the microcanonical ensemble in (72) is rather unusual and deserves some discussion. Through the saddle-point relation (72), we have defined a temperature by enforcing the known values of energy and momentum of the cluster to be such that they can easily be recognized as the average energy and momentum in the canonical ensemble, that is, in the cluster's rest frame where $P = (M, \mathbf{0})$ and $\beta = (\bar{\beta}, \mathbf{0})$:

$$M = -\frac{\partial}{\partial \bar{\beta}} \log Z(\bar{\beta}, \mathbf{Q}), \quad (77)$$

with $T = 1/\bar{\beta}$. On the other hand, it is also possible [18] to extend the relation

$$\frac{1}{T} \equiv \frac{\partial S}{\partial M} \quad \text{with} \quad S = \log \Omega \quad (78)$$

to the microcanonical regime. This definition gives rise to the following equation, by using (69):

$$\frac{1}{T} = \frac{1}{\Omega} \lim_{\varepsilon \rightarrow 0} \frac{1}{(2\pi i)^4} \int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} d^4 z z^0 e^{M z^0} Z(z, \mathbf{Q}), \quad (79)$$

which implies a different definition of temperature with respect to (77). At the leading order of the asymptotic expansion of the above integral and Ω , the previous equation reads

$$\frac{1}{T} \simeq \frac{\bar{\beta}' \exp \left[M + \frac{\partial}{\partial \bar{\beta}'} \log Z(\bar{\beta}', \mathbf{Q}) \right] (C_V(\bar{\beta}) \bar{\beta}^2)^{-1/2}}{\exp \left[M + \frac{\partial}{\partial \bar{\beta}} \log Z(\bar{\beta}, \mathbf{Q}) \right] (C_V(\bar{\beta}') \bar{\beta}'^2)^{-1/2}}, \quad (80)$$

where $\bar{\beta}$ is the solution of (77) and $\bar{\beta}'$ the one of

$$\frac{1}{\bar{\beta}'} + M = -\frac{\partial}{\partial \bar{\beta}'} \log Z(\bar{\beta}', \mathbf{Q}). \quad (81)$$

If the system is very large, i.e. in the thermodynamical limit, the temperature $1/\bar{\beta}'$ is much less than M so, according to (81) $\bar{\beta}' \simeq \bar{\beta}$, $1/T \simeq \bar{\beta}' \simeq \bar{\beta}$ and the two definitions coincide, as expected.

It is worth pointing out that, even in the canonical ensemble, for finite volumes, $\log Z$ is not a linear function of V (see (70)) and this has the remarkable consequence that T , in both definitions, is not a function of M/V but of M and V separately. Otherwise stated, if T and V are used as independent thermodynamical parameters, the mean energy is not an extensive variable as it does not scale linearly with V . Of course, this mostly unfamiliar feature disappears in the thermodynamic limit.

6 Physical observables

The comparison of the model predictions with experimental measurements involves the calculation of quantities which can always be written as averages or expectation values of some operator. For instance, the average multiplicity of the j th hadron species in the grand-microcanonical ensemble can be written as

$$\langle \hat{N}_j \rangle = \frac{\sum_{\{N_j\}} N_j \Omega_{\{N_j\}}}{\Omega}, \quad (82)$$

the correlations between j th and k th hadron species as the expectation value of $(\hat{N}_j - \langle N_j \rangle)(\hat{N}_k - \langle N_k \rangle)$ and the probability of a single configuration $\{N_j\}$ as the expectation value of $\delta_{\hat{N}_1, N_1} \cdots \delta_{\hat{N}_K, N_K}$. The analytical expressions of sums like that in (82) can be obtained by multiplying $\Omega_{\{N_j\}}$ by a factor (a fictitious fugacity) λ_j to the power N_j and taking the derivative with respect to λ_j for $\lambda_j = 1$. Therefore, for the average multiplicity of the j th hadron species:

$$\begin{aligned} \langle \hat{N}_j \rangle &= \frac{\sum_{\{N_j\}} N_j \Omega_{\{N_j\}}}{\Omega} \\ &= \frac{\partial}{\partial \lambda_j} \log \sum_{\{N_j\}} \lambda_j^{N_j} \Omega_{\{N_j\}} \Bigg|_{\lambda_j=1}. \end{aligned} \quad (83)$$

The sum on the right hand side can be generalized to all species:

$$G(\lambda_1, \dots, \lambda_K) = \sum_{\{N_j\}} \Omega_{\{N_j\}} \prod_j \lambda_j^{N_j}, \quad (84)$$

and G can be properly defined as the generating function of the multiparticle multiplicity distribution. Note that $G(1) = \Omega$.

The main advantage of this method of expressing expectation values is that the generating function can be calculated analytically. By using the expression of $\Omega_{\{N_j\}}$ in (65), the right hand side of (84) can be turned into

$$G(\lambda_1, \dots, \lambda_K) = \frac{1}{(2\pi)^4} \int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} d^4 y e^{iP \cdot y} \quad (85)$$

$$\times \exp \left[\sum_j \frac{(2J_j + 1)V}{(2\pi)^3} \int d^3\mathbf{p} \log(1 \pm \lambda_j e^{-ip \cdot y})^{\pm 1} \right]$$

and similarly in the canonical case. Now the expectation value of any operator can be calculated from the generating function by applying many times the differential operators $D_j = \lambda_j \partial / \partial \lambda_j$. In fact, according to (84), for the M th power of N_j :

$$\langle \hat{N}_j^M \rangle = \frac{1}{\Omega} \left[\prod_{i=1}^M \lambda_j \frac{\partial}{\partial \lambda_j} \right] G(\lambda_1, \dots, \lambda_K) \Big|_{\lambda=1}. \quad (86)$$

Then, since the operators D_j and D_k commute, we can write formally, for any function of N_1, \dots, N_K :

$$\langle F(\{\hat{N}_j\}) \rangle = \frac{1}{\Omega} F(D_1, \dots, D_K) G(\lambda_1, \dots, \lambda_K) \Big|_{\lambda=1}. \quad (87)$$

Thereby analytical expressions of various observables can be inferred. For instance, by using (86) with G given by (85), the average multiplicity of the j th hadron, in the Boltzmann statistics limit, turns out to be

$$\langle \hat{N}_j \rangle = \frac{1}{\Omega(P)} \frac{(2J_j + 1)V}{(2\pi)^3} \int d^3\mathbf{p} \Omega(P - p_j), \quad (88)$$

where $\Omega(P)$ is given by (53). It is worth remarking that, since $\Omega(P - p_j)$ vanishes when $(P - p_j)^2 < 0$, the integration in momentum is cut off when, in the cluster's rest frame, the energy of the particle exceeds the cluster's mass, as it should naturally occur in a microcanonical framework.

Despite their simple appearance, expressions like (88) are extremely hard to calculate analytically. In fact, the whole issue of providing closed formulae of multiplicities, correlations etc. reduces to the calculation of the generating function in (85). However, an explicit solution of that four-dimensional integral is known only in the two limiting cases of ultrarelativistic (vanishing masses) and non-relativistic gas [19]. For the relativistic gas with massive particles, which pertains to the hadronic system, no closed formula useful for numerical evaluation has ever been obtained, not even as a series. Therefore, the only practicable way of calculating averages within the microcanonical ensemble is to evaluate $\Omega_{\{N_j\}}$ integral expressions like (45) (which is in turn made up of integral terms like (56)) and sum over all possible channels. However, also those integrals have been solved analytically only in the aforementioned two limiting cases, because the functions to be dealt with are essentially the same. Several authors have tried approximations [20] but in most cases it is difficult to keep the error under control so that, at some fixed order truncation of the expansions, the relative accuracy may vary from some percent to a factor of 10 [21]. Thus, the problem of exploring the hadronic microcanonical ensemble can be attacked only numerically through Monte-Carlo integration. This has been done by Werner and Aichelin in a quite recent paper with a method based on the Metropolis algorithm [5]. In the next paper,

we will present a full numerical calculation for the ideal hadron-resonance gas which exploits a modification of that method, very effective for large clusters, taking advantage of the grand-canonical limit of the multiplicity distributions as proposal matrix in the Metropolis algorithm.

7 Summary and outlook

This paper is the first of a series of two devoted to the study of the microcanonical ensemble of the hadron gas, which is the most fundamental framework for the statistical hadronization model. In this work we have mainly developed the analytical formalism, while numerical calculations will be the main subject of the second paper. The main achievements can be summarized as follows.

(1) We have provided a consistent formulation of the statistical hadronization model starting from purposely defined quantum transition probabilities. This formulation is much easier to handle than previous ones based on time-reversal arguments and S -matrix elements averaging and allows one to calculate any final-state observables more straightforwardly. Furthermore, it is easier to extend it to the case of angular momentum and parity conservation, whenever needed. We think that this formulation clarifies once more that it is possible to account for the observed statistical equilibrium of the final-state hadronic multiplicities as a result of prehadronic cluster decays, without invoking a thermalization process driven by collisions between formed hadrons.

(2) We have worked out the rates of exclusive channels neglecting angular momentum, parity, isospin and C -parity conservation (which are important only for very small hadronizing systems) and recovered known expressions in the statistical model. We have obtained an expression of the phase space volume in full quantum statistics as a cluster decomposition, (44), generalizing previous ones [11] which are valid only asymptotically, i.e. in the limit of a very large cluster (in practice with a linear size roughly larger than 3–10 fm). This expression is valid provided that relativistic quantum field effects are neglected, i.e. the hadronizing cluster should be sufficiently larger than the Compton wavelengths of the hadrons.

(3) We have shown analytically how the canonical ensemble can be obtained as an approximation of the microcanonical ensemble for large volumes and mass of the cluster.

In the second forthcoming paper [12], the numerical integration of the microcanonical expressions obtained in this paper will be carried out by means of a Monte-Carlo method. This will enable a detailed comparison with the canonical ensemble and to establish the range of validity of the latter, which has been used in the actual comparisons with measured hadronic multiplicities [1, 7]. Besides, the implementation of a reasonably fast and reliable Monte-Carlo algorithm for the microcanonical hadronization of single clusters in high energy collisions is a decisive step for further tests of the statistical hadronization model.

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Appendices

A Symmetries of operator W

We briefly discuss the requirements on the operator W in (4) for the fulfillment of the known strong-interactions symmetries. If $U(g)$ is the unitary representation onto a Hilbert space of an element g belonging to a symmetry group, then we can write

$$U(g)WU(g)^{-1} = W. \quad (\text{A.1})$$

Thus, as $\hat{\eta}$ depends, by definition, only on Casimir operators, we have

$$U(g)P_VU(g)^{-1} = P_V, \quad \sum_{h_V} |U(g)h_V\rangle\langle U(g)h_V| = \sum_{h_V} |h_V\rangle\langle h_V|. \quad (\text{A.2})$$

Since $U(g)$ is a one-to-one correspondence, the requirement is met if, for any g , every $|U(g)h_V\rangle$ is a multihadronic state of the cluster, i.e. a $|h'_V\rangle$ or a linear combination of them. This is obvious if $|h_V\rangle$ are eigenvectors of $U(g)$, which is the case for the $U(1)$ groups associated with abelian additive charges, and quite straightforward for isospin $SU(2)$ since P_V is the projector identity as far as the isospin degrees of freedom are concerned; also charge conjugation symmetry is trivially satisfied.

The situation is rather different for space-time symmetries. In this case, the translation, rotation and reflection operators transform the projector P_V in the projector onto the translated, rotated or reflected cluster respectively; only if this object is the same as the starting one, symmetry is fulfilled. Therefore, angular momentum and parity are conserved only if the cluster is spherical in shape, while energy and momentum are not conserved because of the finite volume.

B Decomposition of the Poincaré group projector

The general transformation of the extended Poincaré group g_z may be factorized as

$$g_z = T(x)Z\Lambda = T(x)ZL_{\hat{n}}(\xi)R, \quad (\text{B.1})$$

where $T(x)$ is a translation by the four-vector x , $Z = I, \Pi$ is either the identity or the space inversion and $\Lambda = L_{\hat{n}}(\xi)R$ is a general orthochronous Lorentz transformation written as the product of a boost of a hyperbolic angle ξ along the spacelike axis \hat{n} and a rotation R depending on three Euler angles. Thus (7) becomes

$$P_{P,J,\lambda,\pi}$$

$$\begin{aligned} &= \frac{1}{2} \sum_{Z=I,\Pi} \frac{\dim \nu}{(2\pi)^4} \int d^4x \int d\Lambda D^\nu(T(x)Z\Lambda)_i^{i*} \\ &\quad \times U(T(x)Z\Lambda) \\ &= \frac{1}{2} \sum_{Z=I,\Pi} \frac{\dim \nu}{(2\pi)^4} \int d^4x \int d\Lambda e^{iP \cdot x} \pi^z D^\nu(\Lambda)_i^{i*} \\ &\quad \times U(T(x))U(Z)U(\Lambda), \end{aligned} \quad (\text{B.2})$$

where $z = 0$ if $Z = I$ and $z = 1$ if $Z = \Pi$. In the above equation, by $d\Lambda$ we meant the invariant normalized measure of the Lorentz group, which can be written as [22]

$$d\Lambda = dL_{\mathbf{n}}(\xi) dR = \sinh^2 \xi d\xi \frac{d\Omega_{\hat{n}}}{4\pi} dR, \quad (\text{B.3})$$

dR being the well known invariant measure of the $SU(2)$ group.

If the initial state $|i\rangle$ has a vanishing momentum, i.e. $P = (M, \mathbf{0})$, then the Lorentz transformation Λ must not involve any non-trivial boost transformation with $\xi \neq 0$ for the matrix element $D^\nu(\Lambda)_i^{i*}$ not to vanish. Therefore Λ reduces to the rotation R and we can write

$$\begin{aligned} P_{P,J,\lambda,\pi} &= \frac{1}{2} \sum_{Z=I,\Pi} \frac{1}{(2\pi)^4} \int d^4x (2J+1) \int dR e^{iP \cdot x} \pi^z \\ &\quad \times D^J(R)_{\lambda}^{\lambda*} U(T(x))U(Z)U(R). \end{aligned} \quad (\text{B.4})$$

Since $[Z, R] = 0$, we can move the $U(Z)$ operator to the right of $U(R)$ and recast the above equation as

$$\begin{aligned} P_{P,J,\lambda,\pi} &= \frac{1}{(2\pi)^4} \int d^4x e^{iP \cdot x} U(T(x)) \\ &\quad \times (2J+1) \int dR D^J(R)_{\lambda}^{\lambda*} U(R) \frac{1 + \pi U(\Pi)}{2}, \end{aligned} \quad (\text{B.5})$$

which is (8).

C Proof of equations (18) and (28)

We shall prove (18) in non-relativistic quantum mechanics. It is assumed that $|\mathbf{k}\rangle$ is a complete set of states in a region A with volume V , with eigenfunctions $\psi_{\mathbf{k}}(\mathbf{r})$ and that the transformation from σ to τ polarization states is unitary. Thus

$$\langle \mathbf{r}\sigma | \mathbf{k}\tau \rangle = \begin{cases} \psi_{\mathbf{k}}(\mathbf{r}) U_{\sigma\tau} & \text{if } \mathbf{r} \in A, \\ 0 & \text{if } \mathbf{r} \notin A, \end{cases} \quad (\text{C.1})$$

where $U_{\sigma\tau}$ is the element of a unitary matrix. Hence

$$\begin{aligned} \sum_{\mathbf{k},\tau} |\langle \mathbf{p}\sigma | \mathbf{k}\tau \rangle|^2 &= \sum_{\mathbf{k},\tau} \langle \mathbf{p}\sigma | \mathbf{k}\tau \rangle \langle \mathbf{k}\tau | \mathbf{p}\sigma \rangle \\ &= \sum_{\mathbf{k},\tau} \int_A d^3\mathbf{r} \int_A d^3\mathbf{r}' \langle \mathbf{p}\sigma | \mathbf{r}\sigma \rangle \langle \mathbf{r}\sigma | \mathbf{k}\tau \rangle \langle \mathbf{k}\tau | \mathbf{r}'\sigma \rangle \langle \mathbf{r}'\sigma | \mathbf{p}\sigma \rangle \\ &= \sum_{\mathbf{k},\tau} \int_A d^3\mathbf{r} \int_A d^3\mathbf{r}' \frac{e^{i\mathbf{p} \cdot (\mathbf{r}' - \mathbf{r})}}{(2\pi)^3} \psi_{\mathbf{k}}(\mathbf{r}) \psi_{\mathbf{k}}^*(\mathbf{r}') |U_{\sigma\tau}|^2, \end{aligned} \quad (\text{C.2})$$

where we have used the normalization of the states $\langle \mathbf{p} | \mathbf{p}' \rangle = \delta^3(\mathbf{p} - \mathbf{p}')$. Since the $\psi_{\mathbf{k}}$ are a complete set of eigenfunctions in A :

$$\sum_{\mathbf{k}} \psi_{\mathbf{k}}(\mathbf{r}) \psi_{\mathbf{k}}^*(\mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}'); \quad (\text{C.3})$$

thus, taking into account that U is unitary, (C.2) turns into

$$\sum_{\mathbf{k}, \tau} |\langle \mathbf{p} \sigma | \mathbf{k} \tau \rangle|^2 = \frac{1}{(2\pi)^3} \int_A d^3\mathbf{r} \int_A d^3\mathbf{r}' \delta^3(\mathbf{r} - \mathbf{r}') = \frac{V}{(2\pi)^3}, \quad (\text{C.4})$$

Q.E.D.

Likewise, (28) can be proved by calculating

$$\begin{aligned} & \sum_{\mathbf{k}, \tau} \langle \mathbf{p}_1 \sigma_1 | \mathbf{k} \tau \rangle \langle \mathbf{k} \tau | \mathbf{p}_2 \sigma_2 \rangle \\ &= \sum_{\mathbf{k}, \tau} \int_A d^3\mathbf{r} \int_A d^3\mathbf{r}' \\ & \quad \times \langle \mathbf{p}_1 \sigma_1 | \mathbf{r} \sigma_1 \rangle \langle \mathbf{r} \sigma_1 | \mathbf{k} \tau \rangle \langle \mathbf{k} \tau | \mathbf{r}' \sigma_2 \rangle \langle \mathbf{r}' \sigma_2 | \mathbf{p}_2 \sigma_2 \rangle \\ &= \sum_{\mathbf{k}, \tau} \int_A d^3\mathbf{r} \int_A d^3\mathbf{r}' \frac{e^{i\mathbf{p}_2 \cdot \mathbf{r}' - i\mathbf{p}_1 \cdot \mathbf{r}}}{(2\pi)^3} \\ & \quad \times \psi_{\mathbf{k}}(\mathbf{r}) \psi_{\mathbf{k}}^*(\mathbf{r}') U_{\sigma_1 \tau} U_{\sigma_2 \tau}^* \\ &= \delta_{\sigma_1, \sigma_2} \frac{1}{(2\pi)^3} \int_A d^3\mathbf{r} e^{i\mathbf{r} \cdot (\mathbf{p}_2 - \mathbf{p}_1)}, \end{aligned} \quad (\text{C.5})$$

where (C.3) and unitarity of U have been used.

D Extra strangeness suppression

The use of an extra strangeness suppression parameter γ_S is quite common in statistical model analyses in canonical and grand-canonical ensembles. We show here how this parameter can be inserted in the microcanonical ensemble giving rise to the usual formulae in the large volume limit. All that is needed is to multiply W by an operator which adds a factor γ_S for each *pair* of valence strange quarks which is created or destroyed in the final state. Thus (3) becomes

$$\Gamma_f \rightarrow \Gamma_f \gamma_S^{|N_{Sf} - N_{Si}|}, \quad (\text{D.1})$$

where N_{Si} the number of strange quarks in the initial state and $N_{Sf} = \sum_j N_j s_j$ the one in the final state, s_j being the number of valence strange quarks in the j th hadron species. Then it is quite straightforward to extend the formulae shown in this paper for the presence of this additional factor. In particular, if $N_{Si} = 0$, the rate can be written

$$\Gamma_f = |\eta_i|^2 \Omega'_{\{N_j\}}, \quad (\text{D.2})$$

where

$$\Omega'_{\{N_j\}} = \gamma_S^{\sum_j N_j s_j} \Omega_{\{N_j\}} \quad (\text{D.3})$$

and $\Omega_{\{N_j\}}$ as quoted throughout the paper. As far as mesons with fractional content $C_S \in [0, 1]$ of $\langle s\bar{s} \rangle$ are concerned (η for instance), an independent incoherent superposition of

the rates is assumed as though the meson was a $\langle s\bar{s} \rangle$ state in a fraction C_S of observed reactions. Therefore (D.3) must be rewritten as

$$\Omega'_{\{N_j\}} = \prod_j f_j^{N_j} \Omega_{\{N_j\}}, \quad (\text{D.4})$$

where

$$f_j = \begin{cases} 1 - C_{Sj} + C_{Sj} \gamma_S^2 & \text{for unflavored mesons.} \\ \gamma_S^{s_j} & \text{otherwise.} \end{cases} \quad (\text{D.5})$$

These modifications lead to a corresponding modification of the microcanonical partition function:

$$\Omega' = \sum_{\{N_j\}} \Omega'_{\{N_j\}} = \sum_{\{N_j\}} f_j^{N_j} \Omega_{\{N_j\}}. \quad (\text{D.6})$$

Under the same condition of the validity of the approximations, it can be proved, going along the equations quoted in Sect. 4, that this expression can be calculated explicitly. The grand-microcanonical partition function reads

$$\begin{aligned} \Omega &= \frac{1}{(2\pi)^4} \int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} d^4y e^{iP \cdot y} \\ & \times \exp \left[\sum_j \frac{(2J_j + 1)V}{(2\pi)^3} \int d^3\mathbf{p} \log(1 \pm f_j e^{-i\mathbf{p} \cdot y})^{\pm 1} \right]. \end{aligned} \quad (\text{D.7})$$

The canonical partition function can be obtained starting from the above microcanonical partition function like in Sect. 5:

$$\begin{aligned} Z(\beta, \mathbf{Q}) &= \frac{1}{(2\pi)^M} \int_{-\pi}^{+\pi} d^M \phi e^{i\mathbf{Q} \cdot \phi} \\ & \times \exp \left[\sum_j \frac{(2J_j + 1)V}{(2\pi)^3} \int d^3\mathbf{p} \log(1 \pm f_j e^{-\beta \cdot p - i\mathbf{q}_j \cdot \phi})^{\pm 1} \right], \end{aligned} \quad (\text{D.8})$$

which is the same as the one usually employed to derive the hadron multiplicities [1].

References

1. F. Becattini, Z. Phys. C **69**, 485 (1996); F. Becattini, Proceedings of XXXIII Eloisatron Workshop on Universality Features in Multihadron Production and the leading effect (1996) 74, hep-ph/9701275; F. Becattini, U. Heinz, Z. Phys. C **76**, 269 (1997)
2. U. Heinz, Nucl. Phys. A **661**, 140 (1999); R. Stock, Phys. Lett. B **456**, 277 (1999); J. Hormuzdiar et al., McGill preprint MCGILL-00-04, OITS-686, nucl-th/0001044; H. Satz, Nucl. Phys. Proc. Suppl. **94**, 204 (2001); V. Koch, Proceedings of Quark Matter 2002, Nantes, France, 18–24 July 2002, nucl-th/0210070
3. F. Becattini, U. Heinz, Z. Phys. C **76**, 269 (1997); U. Heinz, Nucl. Phys. A **661**, 140 (1999); R. Stock, Phys. Lett. B **456**, 277 (1999)
4. R. Stock, hep-ph/0212287
5. K. Werner, J. Aichelin, Phys. Rev. C **52**, 1584 (1995)

6. F.M. Liu, K. Werner, J. Aichelin, *Phys. Rev. C* **68**, 024905 (2003)
7. F. Becattini, G. Passaleva, *Eur. Phys. J. C* **23**, 551 (2002)
8. E. Fermi, *Progr. Th. Phys.* **5**, 570 (1950)
9. R. Hagedorn, *N. Cim.* **15**, 434 (1960)
10. F. Cerulus, *N. Cim.* **22**, 958 (1961)
11. M. Chaichian, R. Hagedorn, M. Hayashi, *Nucl. Phys. B* **92**, 445 (1975)
12. F. Becattini, L. Ferroni, Statistical hadronization and hadronic microcanonical ensemble II, in preparation
13. A. Chodos et al., *Phys. Rev. D* **9**, 3471 (1974)
14. G. Marchesini, B.R. Webber et al., *Comp. Phys. Comm.* **67**, 465 (1992).
15. R. Hagedorn, CERN lectures Thermodynamics of strong interactions (1970); R. Hagedorn, CERN-TH 7190/94, in *Hot Hadronic Matter* (1994) 13
16. W. Blumel, P. Koch, U. Heinz, *Z. Phys. C* **63**, 637 (1994)
17. H. Weyl, *The theory of groups and quantum mechanics*
18. D.H. Gross, *Eur. Phys. J. B* **15**, 115 (2000) and references therein
19. J.V. Lepore, R.N. Stuart, *Phys. Rev.* **94**, 1724 (1954)
20. R. Milburn, *Rev. Mod. Phys.* **27**, 1 (1955) and references therein; S. Belenkij et al., *Usp. Fiz. Nauk.* **62**, 1 (1957) and references therein; G. Fialho, *Phys. Rev.* **105**, 328 (1957)
21. F. Cerulus, R. Hagedorn, *Suppl. N. Cim. IX, serie X*, **2**, 646 (1958)
22. W.K. Tung, *Group theory in physics* (World Scientific, Singapore); W. Ruhl, *The Lorentz group and harmonic analysis* (A. Benjamin, New York 1970)